ON THE HARDNESS OF THE MODULE LEARNING WITH ERRORS PROBLEM

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Context in public key cryptography

→ Need for alternatives

- Post-quantum secure,
- Efficient,
- New functionalities, different types of constructions.

NIST competition

- Lattice-based
  3 over 4 standards
- Code-based
- Multivariate, Isogenies, Hash based ..

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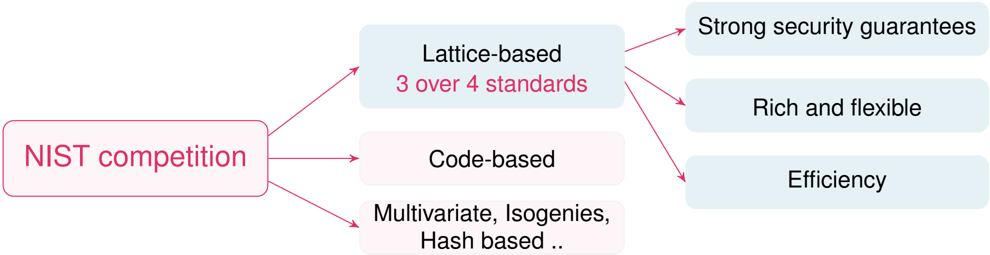

Lattices

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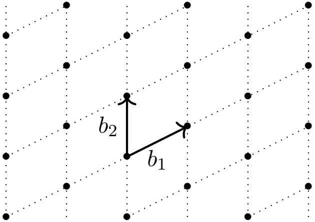


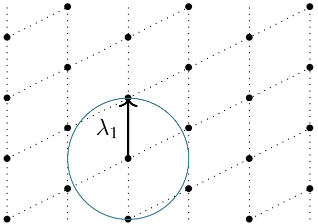
Lattice

$\mathcal{L}(\mathbf{B}) = \{\sum_{1=i}^{n} a_i \mathbf{b}_i, a_i \in \mathbb{Z}\}$, where the $(\mathbf{b}_i)_{1 \le i \le n}$'s, linearly independent vectors, are a basis of $\mathcal{L}(\mathbf{B})$.

Shortest Vector Problem (SVP)

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n:

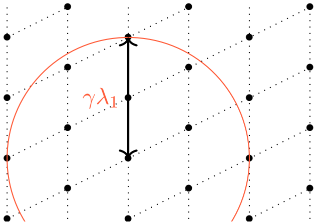
Output: find the shortest non-zero vector $\mathbf{x} \in \mathcal{L}(\mathbf{B})$.



Approx Shortest Vector Problem (Approx SVP$_\gamma$)

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n:

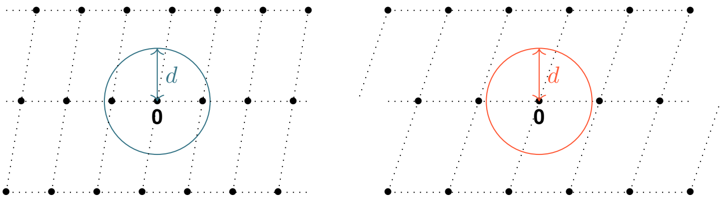
Output: find a non-zero vector $\mathbf{x} \in \mathcal{L}(\mathbf{B})$ such that $\|\mathbf{x}\| \leq \gamma\lambda_1(\mathcal{L}(\mathbf{B}))$



Gap Shortest Vector Problem (GapSVP)

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n and $d > 0$:

Output:
- YES: there is $\mathbf{z} \in \mathcal{L}(\mathbf{B})$ non-zero such that $\|\mathbf{z}\| < d$,
- NO: for all non-zero vectors $\mathbf{z} \in \mathcal{L}(\mathbf{B})$: $\|\mathbf{z}\| \geq d$.



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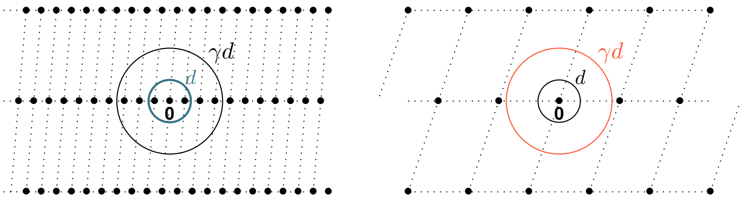
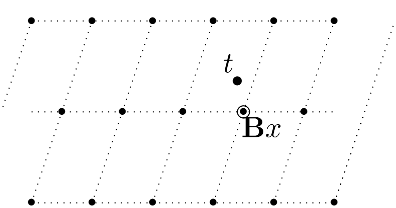


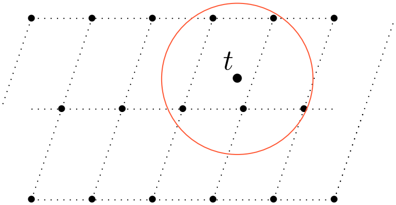
Closest Vector Problem

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n and $\mathbf{t} \in \mathbb{Z}^m$:

Output: find $\mathbf{x} \in \mathbb{Z}^n$ minimizing $||\mathbf{Bx} - \mathbf{t}||$.

Approx variant: find $\mathbf{x} \in \mathbb{Z}^n$ such that $||\mathbf{Bx} - \mathbf{t}|| \leq \gamma \cdot \mathrm{dist}(\mathbf{t}, \Lambda(\mathbf{B}))$.





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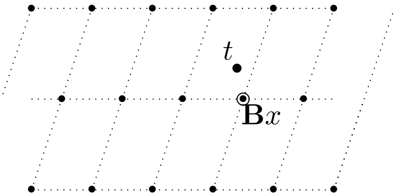
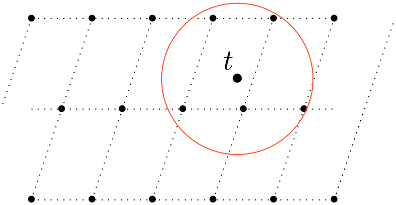
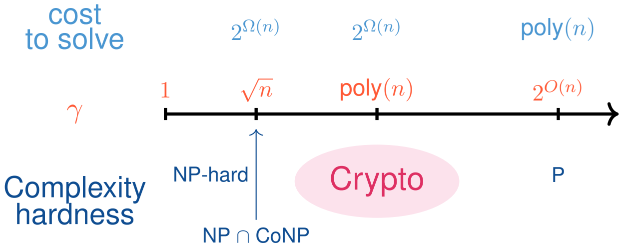




How hard is it to solve those problems?

Hardness of Approx SVP$_\gamma$



Conjecture

There is no polynomial time algorithm that approximates this lattice problem and its variants to within polynomial factors.

At the heart of lattice-based cryptography
the Learning With Errors problem

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- Introduced by Regev in 2005

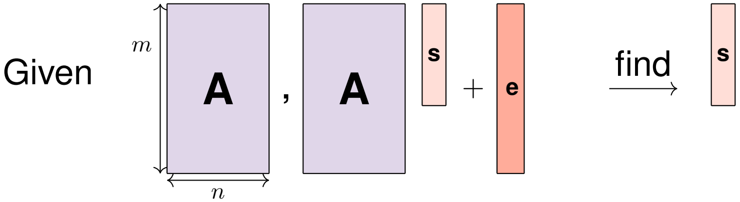
Problem: solve a linear system with noise.

Find $(s_1, s_2, s_3, s_4, s_5)$ such that:

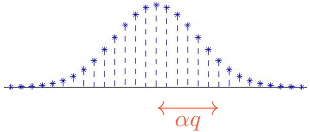
$$\begin{array}{rcrl}
s_1 + 22s_2 + 17s_3 + 2s_4 + s_5 & \approx & 16 & \mod 23 \\
3s_1 + 2s_2 + 11s_3 + 7s_4 + 8s_5 & \approx & 17 & \mod 23 \\
15s_1 + 13s_2 + 10s_3 + 3s_4 + 5s_5 & \approx & 3 & \mod 23 \\
17s_1 + 11s_2 + 20s_3 + 9s_4 + 3s_5 & \approx & 8 & \mod 23 \\
2s_1 + 14s_2 + 13s_3 + 6s_4 + 7s_5 & \approx & 9 & \mod 23 \\
4s_1 + 21s_2 + 9s_3 + 5s_4 + s_5 & \approx & 18 & \mod 23 \\
11s_1 + 12s_2 + 5s_3 + s_4 + 9s_5 & \approx & 7 & \mod 23
\end{array}$$

$\rightsquigarrow$ With an arbitrary number of equations.

LWE_q^n



- $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m \times n})$,
- $\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$,
- e small compared to q.



Discrete Gaussian error $D_{\mathbb{Z},\alpha q}$

Search version: Given $(\mathbf{A}, \mathbf{b} = \mathbf{As} + \mathbf{e})$, find $\mathbf{s}$.
Decision version: Distinguish from $(\mathbf{A}, \mathbf{b})$ with $\mathbf{b}$ uniform.

Solving LWE

- Exhaustive search
  - Try all the $\mathbf{s} \in \mathbb{Z}_q^n \to$ is $\mathbf{b} - \mathbf{A}\mathbf{s}$ small?
  - $\Rightarrow$ cost around q^n.

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  - Other possibility: guess the n first errors, find $\mathbf{s}$ $\rightarrow$ is $\mathbf{b} - \mathbf{A}\mathbf{s}$ small?
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- How to do better?
  - LWE is a lattice problem: consider

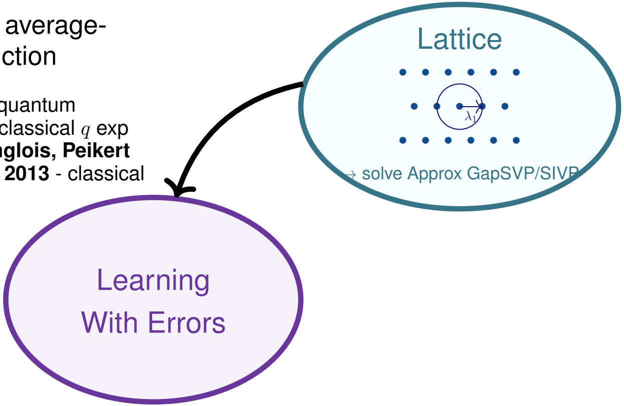
    $$\Lambda_q(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Z}^m : \mathbf{y} = \mathbf{A}\mathbf{s} \bmod q \text{ for } \mathbf{s} \in \mathbb{Z}^n\}.$$

    Solving LWE $\Leftrightarrow$ solving CVP in this lattice.
  - Cost: $\left(\frac{n \log q}{\log^2 \alpha}\right)^{\frac{n \log q}{\log^2 \alpha}}$.

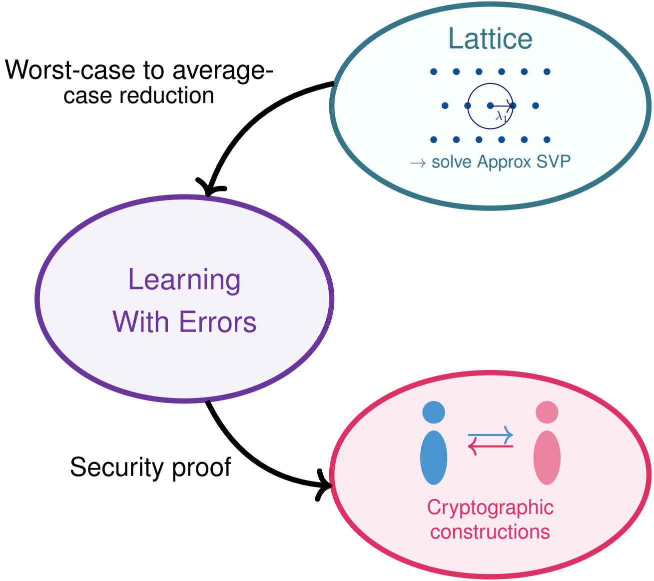
Hardness of the Learning With Errors problem

Worst-case to average-case reduction

- **Regev 2005** - quantum
- **Peikert 2009** - classical q exp
- **Brakerski, Langlois, Peikert Regev, Stehlé 2013** - classical



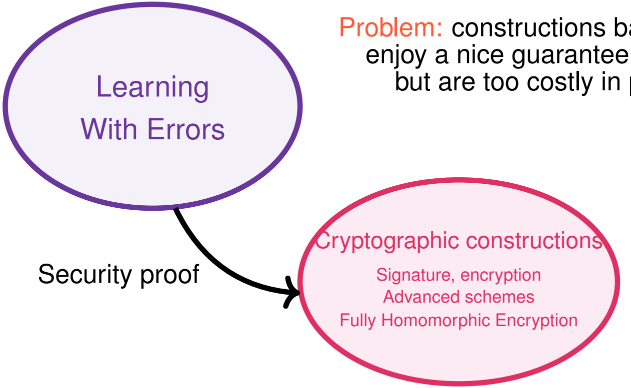
Using LWE to build provable constructions - theory



Using LWE

Hardness of LWE used as a foundation for many constructions.

Problem: constructions based on LWE enjoy a nice guarantee of security but are too costly in practice.



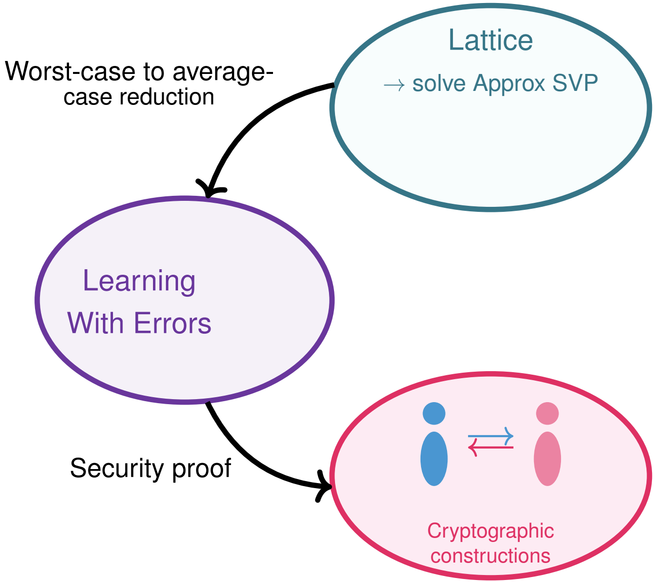
Solutions used today?

Lattice-based NIST finalists

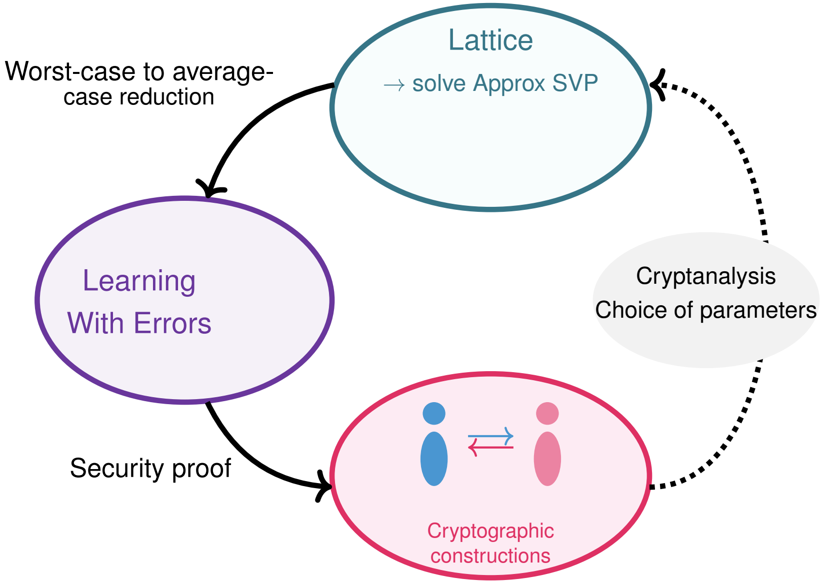
Among the 5 lattice-based finalists, 3 of them are based on (possibly structured) variants of LWE.

- Public Key Encryption
  - **Crystals - Kyber**: Module-LWE with both secret and noise chosen from a centered binomial distribution.
  - **Saber:** Module-LWR (deterministic variant).
  - **NTRU**
  - **FrodoKEM** (as alternate candidate): LWE but with smaller parameters.
- Signature
  - **Crystals - Dilithium**: Module-LWE with both secret and noise chosen in a small uniform interval, and Module-SIS.
  - **Falcon**: Ring-SIS on NTRU matrices.

Using LWE to build constructions



Using LWE to build constructions in practice



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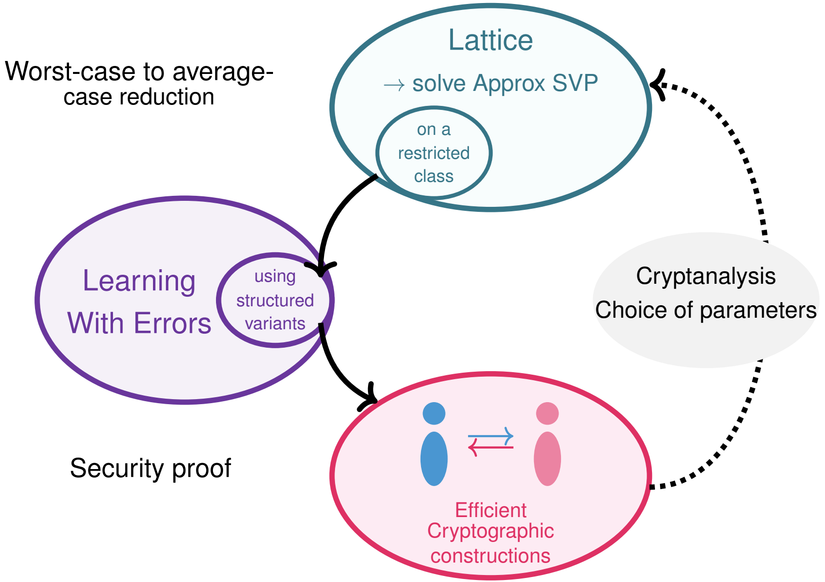


Studying theoretical hardness of variants of LWE

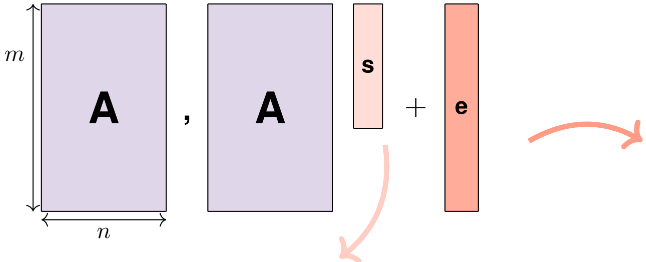
$\rightarrow$ A better understanding of the underlying hardness hypothesis
to reduce the gap between what is proven and what is used in practice

- Hardness of LWE variants
  - Using the Rényi divergence in reductions.

- Recent results on the hardness of Module-LWE
  - Binary (bounded) secret,
  - Classical hardness,
  - Entropic secret.

LWE variants

Choose another distribution for the secret or the error.
Regev 2009: uniform secret and gaussian error.



- Gaussian (continue, discretize, discrete ...),
- Uniform in small interval,
- Binary under conditions.

- Same distribution as the error: in particular Gaussian,
- Binary (Unif in $\{0,1\}^n$),
- Entropic.

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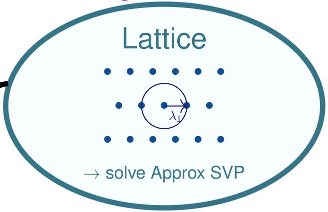


Learning With Errors

- **Peikert 2010** - discrete Gaussian noise
- **Döttling, Müller-Quade 2013** - small uniform
- **Micciancio, Peikert 2013** - small uniform and binary noise
- **Our result 2015 - small uniform, dimension preserving**

Self reductions

- **Applebaum, Cash, Peikert, Sahai 2009** - same error and secret
- **Goldwasser, Kalai, Peikert, Vaikuntanathan 2010** - binary secret
- **Brakerski, Langlois, Peikert, Regev, Stehlé 2013** - binary secret
- **Micciancio 2018** - binary secret
- **Brakerski, Döttling 2020** - entropic secret

Using the Rényi divergence

with S. Bai, T. Lepoint, D. Stehlé, R. Steinfeld and A. Sakzad

- Introduction of RD in security proofs as a measure of distribution closeness,

Let D_1, D_2 be two discrete probability distributions.

Statistical distance

$$\Delta(D_1, D_2) = \frac{1}{2} \sum_{x \in \mathsf{Supp}(D_1)} \left| D_1(x) - D_2(x) \right|,$$

Rényi divergence

$$R_2(D_1, D_2) = \sum_{x \in \mathsf{Supp}(D_1)} \frac{D_1(x)^2}{D_2(x)}.$$

Both fulfill the probability preservation property for an event E:

$$\begin{array}{rcll} D_1(E) - \Delta(D_1, D_2) & \leq & D_2(E) & \text{(additive)} \\ D_1(E)^2 / R_2(D_1, D_2) & \leq & D_2(E) & \text{(multiplicative)} \end{array}$$

Reduction using the Rényi divergence

Reduction from LWE_{ψ_2} (with error ψ_2) to LWE_{ψ_1} (with error ψ_1).

Idea: show that if an adversary can solve LWE_{ψ_1} with a probability of success ε_1 *non negligible*, then he can solve LWE_{ψ_2} with a probability ε_2 non negligible.

Using the probability preservation property, we have that:

$$\begin{array}{lcll} \varepsilon_2 \geq \varepsilon_1 - \Delta(\psi_1, \psi_2) & \Rightarrow & \Delta(\psi_1, \psi_2) & \text{negligible} \\ \varepsilon_2 \geq \varepsilon_1^2 \,/\, R_2(\psi_1, \psi_2) & \Rightarrow & R_2(\psi_1, \psi_2) & \text{constant} \end{array}$$

Note that Rényi Divergence only works for search problems.

Hardness of LWE with small uniform noise

$$
\begin{tikzcd}
\boxed{\text{Decision LWE noise } D_\alpha} \arrow[d, "\psi = D_\alpha + U_\beta"] \\
\boxed{\text{Search LWE noise } \psi} \arrow[d] \\
\boxed{\text{Search LWE noise } U_\beta} \arrow[d] \\
\boxed{\text{Decision LWE noise } U_\beta}
\end{tikzcd}
$$

- Quite direct by adding samples, then decision-to-search reduction. With $(\mathbf{A}, \mathbf{b} = \mathbf{As} + \mathbf{e})$ with $\mathbf{e} \leftarrow D_\alpha$, compute $(\mathbf{A}, \mathbf{b} + \mathbf{e'})$ with $\mathbf{e'} \leftarrow U_\beta$,
- Using that the Rényi divergence $R_2(U_\beta || \psi)$ can be bounded by $1 + 1.05 \cdot \frac{\alpha}{\beta}$.
- Using **Micciancio Mol 11** sample preserving search-to-decision reduction (needs prime q).

More general result

Using the Rényi divergence, we have a reduction:

$$\begin{array}{c}\boxed{\text{Search LWE noise } D_\alpha} \\ \downarrow \\ \boxed{\text{Search LWE noise } \psi}\end{array}$$

- Either $R_2(\psi || D_\alpha)$ is small,
- Either $R_2(\psi || \psi + D_\alpha)$ is small.

- Works nicely if the two distributions are close enough,
- Only needs to compute R_2,
- Distributions may be too far from each other (example: binary).

Studying theoretical hardness of variants of LWE

$\rightarrow$ A better understanding of the underlying hardness hypothesis
to reduce the gap between what is proven and what is used in practice

- Hardness of LWE variants
  - Using the Rényi divergence in reductions.
- Recent results on the hardness of Module-LWE
  - Binary (bounded) secret,
  - Classical hardness,
  - Entropic secret.

Idea: replace $\mathbb{Z}^n$ by $R = \mathbb{Z}[x]/\langle x^n + 1\rangle$

where $n = 2^k$ then the polynomial $x^n + 1$ is irreducible.
Elements of this ring are polynomials of degree less than n.

R is isomorph to $\mathbb{Z}^n$

Let $a \in R$, we have $a(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$,
the isomorphism $R \to \mathbb{Z}^n$ associate

the polynomial $a \in R$ to the vector $\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} \in \mathbb{Z}^n$.

Idea: replace $\mathbb{Z}^n$ by $R = \mathbb{Z}[x]/\langle x^n + 1\rangle$

Let's look at the product of two polynomials $x^n + 1$

- $a(x) = a_0 + a_1 \cdot x + \ldots + a_{n-1} \cdot x^{n-1}$
- $s(x) = s_0 + a_1 \cdot x + \ldots + a_{n-1} \cdot x^{n-1}$

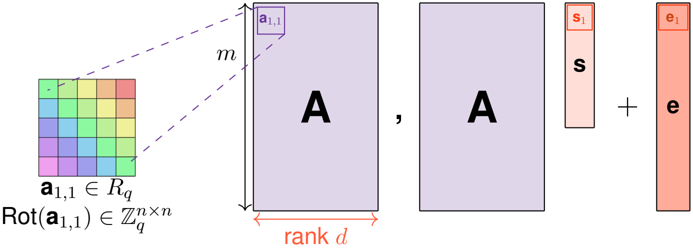
Using matrices, it gives the following block:

$$\begin{bmatrix} a_0 & -a_{n-1} & \cdots & -a_2 & -a_1 \\ a_1 & a_0 & \cdots & -a_3 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_0 & -a_{n-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{bmatrix}$$

Module LWE

Let K be a number field of degree n with R its ring of integers.
Think of K as $\mathbb{Q}[x]/(x^n+1)$ and of R as $\mathbb{Z}[x]/(x^n+1)$ for $n = 2^k$.

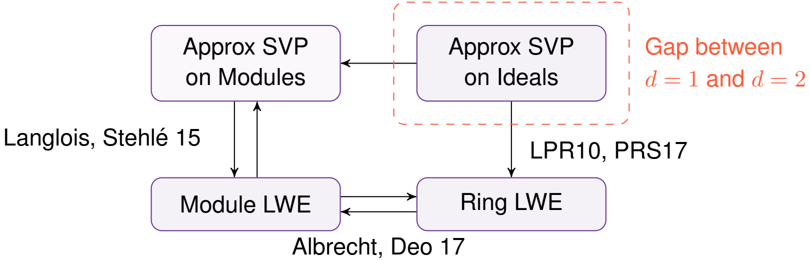
Replace $\mathbb{Z}$ by R, and $\mathbb{Z}_q$ by $R_q = R/qR$.



- $\mathbf{A} \leftarrow U(R_q^{m\times d})$,
- $\mathbf{s} \leftarrow U(R_q^d)$,
- $e \in R^m$ small compared to q.

Special case $d = 1$
is Ring-LWE

Module or Rings?



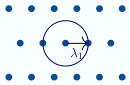
- Choice of parameters
  - $R_q = \mathbb{Z}_q[x]/\langle x^n + 1\rangle$: constraints on parameters $n = 2^k$, $q = 1 \bmod 2n$ ...
- An example of parameters set of a signature:
  - $n = 512 \Rightarrow$ 60 bits of security,
  - $n = 1024 \Rightarrow$ 140 bits of security,
  - $(n = 256, d = 3)$ gives $nd = 768$ which is ”in between”.
- Module LWE allows more flexibility.

Hardness of Module Learning With Errors problem

Worst-case to average-case reduction

- **Langlois Stehlé 2015** - quantum, q poly
- Folklore: adapting **Peikert 2009** gives classical reduction but q exp and only search variant

Module Lattice



$\rightarrow$ solve Module Approx SVP

Module Learning With Errors

An R-module M of rank d defines via the canonical embedding $\sigma : K \rightarrow \mathbb{R}^n$ a module lattice $\sigma(M) \in \mathbb{R}^{nd}$

Self reductions

- **Applebaum, Cash, Peikert, Sahai 2009** - same error and secret

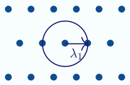
Hardness of Module Learning With Errors problem

with K. Boudgoust, C. Jeudy, W. Wen

Worst-case to average-case reduction

- **Langlois Stehlé 2015** - quantum, q poly
- **Our result**: classical, q poly, decisional but rank linear

Module Lattice



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Self reductions

- **Applebaum, Cash, Peikert, Sahai 2009** - same error and secret
- **Our results 20 & 21**: binary secret, rank inscrease
- **Our result 2023**: η-bounded secret

Hardness of binary Module-LWE

The secret $\mathbf{s} \in R_2^d$ is binary and the secret $\hat{\mathbf{s}} \in R_q^\ell$ is modulo q.

$$m\left\{\underbrace{\mathbf{A}}_{d}\right. , \quad \mathbf{A}\mathbf{s} + \mathbf{e}$$

Module-LWE with binary secret

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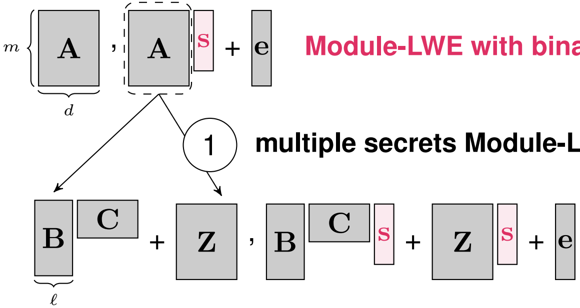

Module-LWE with binary secret

multiple secrets Module-LWE: $\mathbf{A} \approx \mathbf{B}\mathbf{C} + \mathbf{Z}$

(not for Ring-LWE)

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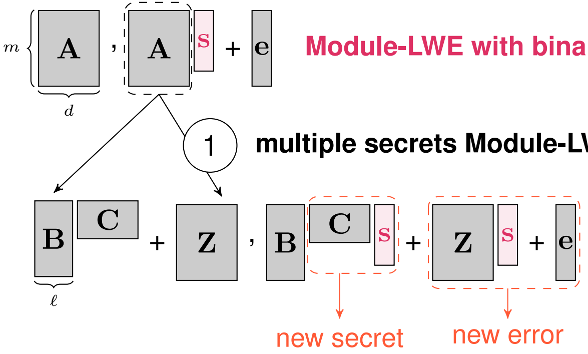

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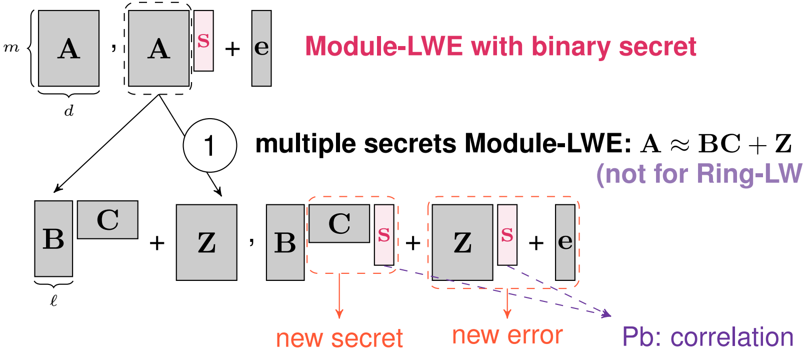

Module-LWE with binary secret

multiple secrets Module-LWE: $\mathbf{A} \approx \mathbf{B}\mathbf{C} + \mathbf{Z}$

(not for Ring-LWE)

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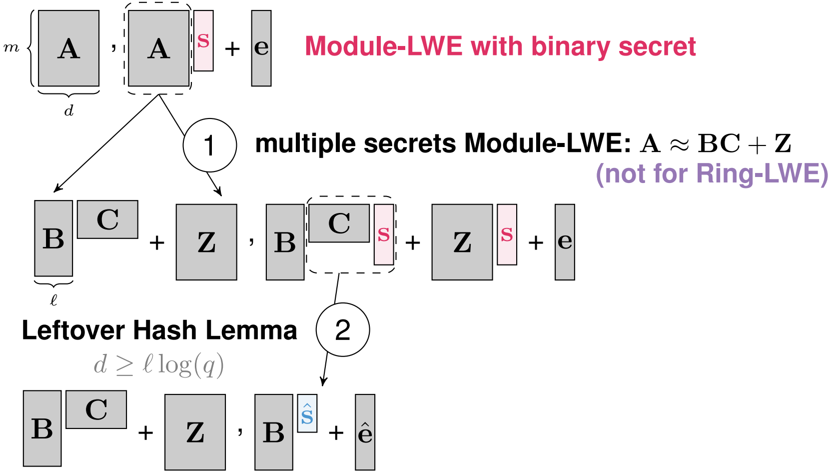

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GREYC
Electronics and Computer Science Laboratory

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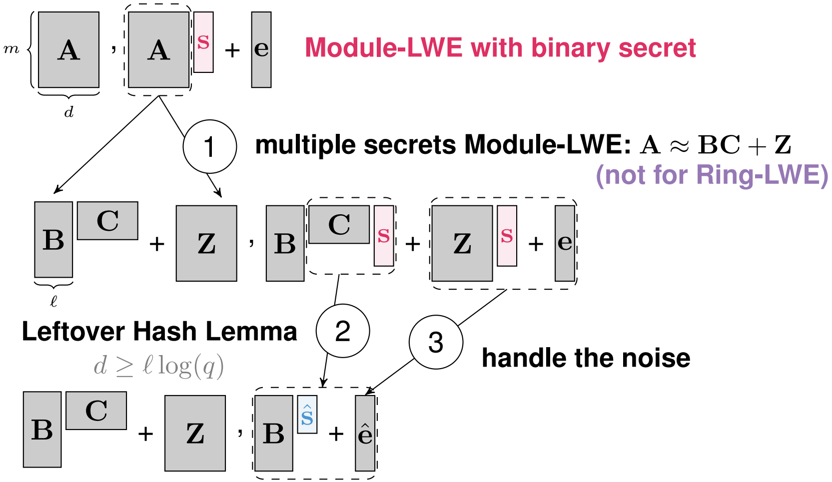

Module-LWE with uniform secret

Hardness of binary Module-LWE

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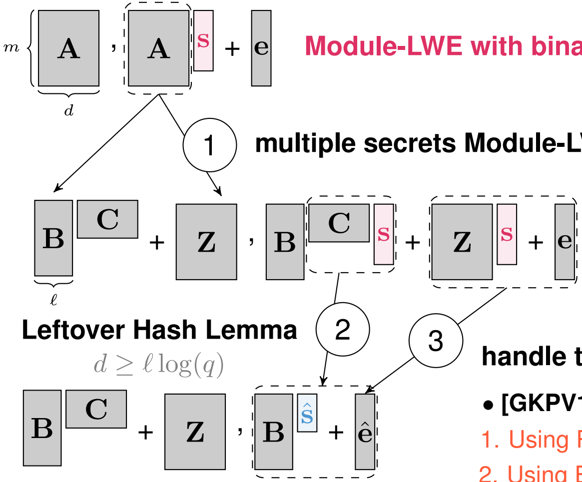

Module-LWE with binary secret

multiple secrets Module-LWE: $\mathbf{A} \approx \mathbf{BC} + \mathbf{Z}$

(not for Ring-LWE)

Leftover Hash Lemma

$d \geq \ell \log(q)$

handle the noise

- **[GKPV10]** noise flooding

1. Using Rényi divergence
2. Using Extended Module-LWE

   additional hint $\mathbf{Zs}$

Module-LWE with uniform secret

Using the Rényi divergence in the reduction

Example: two Gaussians D_β and $D_{\beta,c}$,

$$R_2(D_\beta, D_{\beta,c}) = \exp\left(\frac{2\pi\|c\|^2}{\beta^2}\right)$$

$$\Delta(D_\beta, D_{\beta,c}) = \frac{\sqrt{2\pi}\|c\|}{\beta}$$

With $\|c\| \leq \alpha$

$$\Delta(D_\beta, D_{\beta,c}) = \frac{\sqrt{2\pi}\|c\|}{\beta} \qquad \Rightarrow \alpha/\beta \leq \text{negligible}$$

$$R_2(D_\beta, D_{\beta,c}) = \exp\left(\frac{2\pi\|c\|^2}{\beta^2}\right) \approx 1 + \frac{2\pi\|c\|^2}{\beta^2} \qquad \Rightarrow \alpha/\beta \leq \text{constant}$$

(Taylor expansion at 0)

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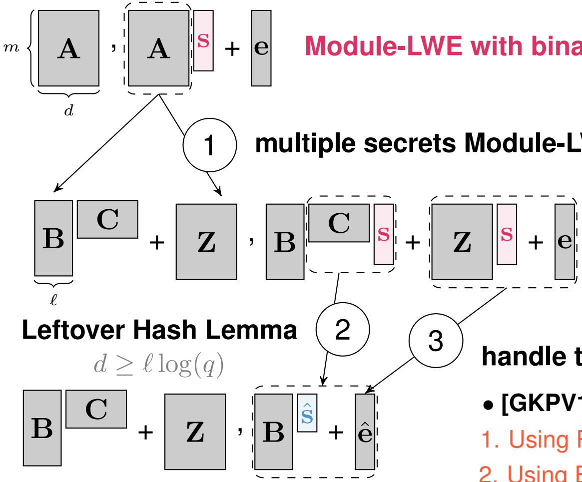

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Module-LWE with uniform secret

Hardness of binary Module-LWE

Standard Module-LWE	$\longrightarrow$	**Binary secret Module-LWE**
modulus q		modulus q
ring degree n		ring degree n
secret $\hat{\mathbf{s}} \in R_q^\ell$		secret $\mathbf{s} \in R_2^d$
Gaussian width α		Gaussian width β
rank ℓ		rank d

Property	Contribution 1	Contribution 2
Minimal rank d	$\ell \log q + O(\log n)$	$(\ell + 1) \log q + \omega(\log n)$
Noise ratio β/α	$O(n^2\sqrt{md})$	$O(n^2\sqrt{d})$
Condition on q	prime	other restrictions
Decision/Search	search	decision

$\rightarrow$ Both proofs have their (dis)advantages

Generalisation to η-bounded secrets

Standard Module-LWE	$\longrightarrow$	η**-Module-LWE**
modulus q		modulus q
ring degree n		ring degree n
secret $\hat{\mathbf{s}} \in R_q^\ell$		secret $\mathbf{s} \in R_\eta^d$
Gaussian width α		Gaussian width β
rank ℓ		rank d

Property	Contribution 1	Contribution 2
Minimal rank d	$\frac{\ell \log q}{\log \eta} + O(\frac{\log n}{\log \eta})$	$\frac{2\ell \log q}{\log \eta} + \omega(\frac{\log n}{\log \eta})$
Noise ratio β/α	$O((\eta - 1)n^2\sqrt{md})$	$O((\eta - 1)^2 n^2 \sqrt{d})$

$\rightarrow$ trade-off between minimal rank and noise ratio

Classical hardness of Module-LWE

$$\begin{tikzcd}
\text{Module-GapSVP}_{\gamma} \arrow[d, "{k \geq 2,\ q \geq 2^{\frac{nk}{2}}}"] \\
\text{Module-LWE}_{n,k,q,\Psi_{\leq \alpha}} \arrow[d, "d \geq 3n + \omega(\log_2 n)"] \\
\text{Binary Module-LWE}_{n,d,q,\Psi_{\leq \kappa}} \arrow[d] \\
\text{Module-LWE}_{n,d,p,\Psi_{\leq \beta}}
\end{tikzcd}$$

- Adapting and merging module variants of **Peikert 09** (classical) and **Peikert, Regev, Stephens-Davidowitz 17** (decisional),
- Adapting **Brakerski, Langlois, Peikert, Regev, Stehlé 13** using Extended Module-LWE,
- Using **Albrecht, Deo 17**, computing bounds on singular values of rotation matrix, loss in the reduction depends on the norm of the secret.

- number theoretic constraints on q
- $d \geq 3n + \omega(\log_2 n)$ and $\beta = \tilde{\Theta}(\frac{n^{5/2}}{\gamma})$

Conclusion

- Hardness of Module-LWE with small secret,
- Hardness of Module-LWE with entropic secret,
- Still conditions on parameters and on the module rank.

Some open questions

- Can we prove those results for smaller rank? In particular Ring-LWE?
- Other error distributions?